

DECOMPOSITION THEOREMS FOR TRIPLE SPACES

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ABSTRACT. A triple space is a homogeneous space G/H where $G = G_0 \times G_0 \times G_0$ is a threefold product group and $H \simeq G_0$ the diagonal subgroup of G . This paper concerns the geometry of the triple spaces with $G_0 = \mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SO}_e(n, 1)$ for $n \geq 2$. We determine the abelian subgroups $A \subset G$ for which there is a polar decomposition $G = KAH$, and we determine for which minimal parabolic subgroups $P \subset G$, the orbit PH is open in G/H .

Date: December 20, 2012.

2000 Mathematics Subject Classification. 22F30, 22E46.

Key words and phrases. triple space, polar decomposition, spherical.

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1. INTRODUCTION

Let G_0 be a real reductive group and let $G = G_0 \times G_0 \times G_0$ and $H = \text{diag}(G_0)$. The corresponding homogeneous space G/H is called a *triple space*. Triple spaces are examples of non-symmetric homogeneous spaces, as there is no involution of G with fixed point group H . It is interesting in the non-symmetric setting to explore properties, which play an important role for the harmonic analysis of symmetric spaces. In this paper we examine the geometric structure of some triple spaces from this point of view.

One important structural result for symmetric spaces is the polar decomposition $G = KAH$. Here $K \subset G$ is a maximal compact subgroup, and $A \subset G$ is abelian. Polar decomposition for a Riemannian symmetric space G/K is due to Cartan, and it was generalized to reductive symmetric spaces in the form $G = KAH$ by Flensted-Jensen [2].

For triple spaces in general, the sum of the dimensions of K , A and H can be strictly smaller than the dimension of G , which obviously prevents $G = KAH$. Here we are interested in the triple spaces with

$$(1.1) \quad G_0 = \text{SL}(2, \mathbb{R}), \text{SL}(2, \mathbb{C}), \text{SO}_e(n, 1) \quad (n = 2, 3, \dots)$$

for which there is no obstruction by dimensions. In Theorem 3.2 we show that indeed these spaces admit a polar decomposition as above, and we determine precisely for which maximal split abelian subgroups A the decomposition is valid. For the simplest choice of group A we describe the indeterminateness of the A -component for a given element in G , and we identify the invariant measure on G/H in these coordinates.

Another important structural result for a Riemannian symmetric space G/K is the fact (closely related to Iwasawa decomposition) that minimal parabolic subgroups P act transitively. For non-Riemannian symmetric spaces there is no transitive action of P , but it is an important result, due to Wolf [7], that P has an orbit on G/H which is open. In Proposition 6.1 we verify that this is the case also for the spaces in (1.1), and we determine precisely for which minimal parabolic subgroups P the orbit through the origin is open.

By combining these results we conclude in Corollary 6.4 that there exist maximal split abelian subgroups A for which $G = KAH$ and for which PH is open for all minimal parabolic subgroups P with $P \supset A$, a property which plays an important role in [5].

An interesting observation (which surprised us) is that in some cases there are also maximal split abelian subgroups A for which PH is open for all minimal parabolic subgroups P with $P \subset A$, but for which the polar decomposition fails (see Remark 6.5).

The fact that the triple space of $\mathrm{SL}(2, \mathbb{C})$ admits open P -orbits follows from [4] p. 152. A homogeneous space of algebraic groups over \mathbb{C} with an open Borel orbit is said to be spherical, cf [1], and the spaces we consider may be seen as prototypes of spherical spaces over \mathbb{R} .

In a final section we introduce an infinitesimal version of the polar decomposition, and show that in the current setting it is valid if and only if the global polar decomposition $G = KAH$ is valid.

The harmonic analysis on $\mathrm{SL}(2, \mathbb{R})$ is an essential example for understanding the harmonic analysis on general reductive groups. We expect the triple spaces considered here to serve similarly for the harmonic analysis on non-symmetric homogeneous spaces, which is yet to be developed.

2. NOTATION

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ be a Cartan decomposition of the Lie algebra \mathfrak{g}_0 of G_0 , and put

$$\mathfrak{k} = \mathfrak{k}_0 \times \mathfrak{k}_0 \times \mathfrak{k}_0, \quad \mathfrak{s} = \mathfrak{s}_0 \times \mathfrak{s}_0 \times \mathfrak{s}_0,$$

then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is also a Cartan decomposition. The maximal abelian subspaces of \mathfrak{s} have the form

$$(2.1) \quad \mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$$

with three maximal abelian subspaces $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ in \mathfrak{s}_0 .

If for each j we let $A_j = \exp \mathfrak{a}_j$ and choose a positive system for the roots of \mathfrak{a}_j , then with $G_0 = K_0 A_j N_j$ for $j = 1, 2, 3$ we obtain the Iwasawa decomposition $G = KAN$ where

$$K = K_0 \times K_0 \times K_0, \quad A = A_1 \times A_2 \times A_3, \quad N = N_1 \times N_2 \times N_3.$$

Likewise we obtain the minimal parabolic subgroup

$$P = P_1 \times P_2 \times P_3 = MAN$$

where $M = M_1 \times M_2 \times M_3$ and each $P_j = M_j A_j N_j$ is a minimal parabolic subgroup of G_0 .

3. POLAR DECOMPOSITION

Let G/H be a homogeneous space of a reductive group G , and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of the Lie algebra of G . A decomposition of G of the form

$$(3.1) \quad G = KAH,$$

with $A = \exp \mathfrak{a}$, for an abelian subspace $\mathfrak{a} \subset \mathfrak{s}$, is said to be a *polar* decomposition. If such a decomposition exists, then the homogeneous space G/H is said to be of *polar type* (see [5]).

The fact that symmetric spaces are of polar type implies in particular that every double space $G/H = (G_0 \times G_0)/\text{diag}(G_0)$ with G_0 a real reductive group admits a polar decomposition. Here we can take

$$\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_0$$

for a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{s}_0$ (in fact, it would suffice to take already the antidiagonal of $\mathfrak{a}_0 \times \mathfrak{a}_0$). Then A has the form $A_1 \times A_2$ with $A_1 = A_2$. In contrast, triple spaces do not admit $G = KAH$ for $A = A_1 \times A_2 \times A_3$ if $A_1 = A_2 = A_3$:

Lemma 3.1. *Let G/H be the triple space of a non-compact semisimple Lie group G_0 . Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be maximal abelian and let $A = A_0 \times A_0 \times A_0$. Then KAH is a proper subset of G .*

Proof. Let $a_0 \in A_0$ be a regular element. We claim that a triple $(g_1, g_2, g_3) = (g_1, a_0, e)$ belongs to KAH only if $g_1 \in K_0 A_0$. Assume $g_i = k_i a_i g$ for $i = 1, 2, 3$ with $k_i \in K_0$, $A_i \in A_0$ and $g \in G_0$. From

$$a_0 = g_2 g_3^{-1} = k_2 a_2 a_3^{-1} k_3^{-1}$$

we deduce that $k_2 = k_3$, and from the regularity of a_0 we then deduce that k_3 belongs to the normalizer $N_{K_0}(\mathfrak{a}_0)$ (see [3], Thm. 7.39). Then

$$g_1 = g_1 g_3^{-1} = k_1 a_1 a_3^{-1} k_3^{-1} \in K_0 A_0.$$

The lemma follows immediately. \square

It was observed in [5] that the triple spaces for the groups considered in (1.1) are of polar type. In the following theorem we determine, for these groups, all the maximal abelian subspaces \mathfrak{a} of \mathfrak{g} for which (3.1) holds.

Theorem 3.2. *Let G_0 be one of groups (1.1) and $\mathfrak{a} \subset \mathfrak{s}$ as in (2.1). Then $G = KAH$ if and only if $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$ has dimension two in \mathfrak{g}_0 . In particular, G/H is of polar type.*

We shall approach $G = KAH$ by a geometric argument. Let $Z_0 = G_0/K_0$ be the Riemannian symmetric space associated with G_0 , and let $z_0 = eK_0 \in Z_0$ denote its origin. Recall that (up to covering) G_0 is the identity component of the group of isometries of Z_0 . Then it is easily seen that $G = KAH$ is equivalent to the following:

Property 3.3. *For every triple (z_1, z_2, z_3) of points $z_j \in Z_0$ there exist a triple (y_1, y_2, y_3) of points $y_j \in Z_0$ with $y_j \in A_j z_0$ for each j , and an isometry $g \in G_0$ such that $gz_j = y_j$ for $j = 1, 2, 3$.*

In order to illustrate the idea of proof, let us first state and prove a Euclidean analogue.

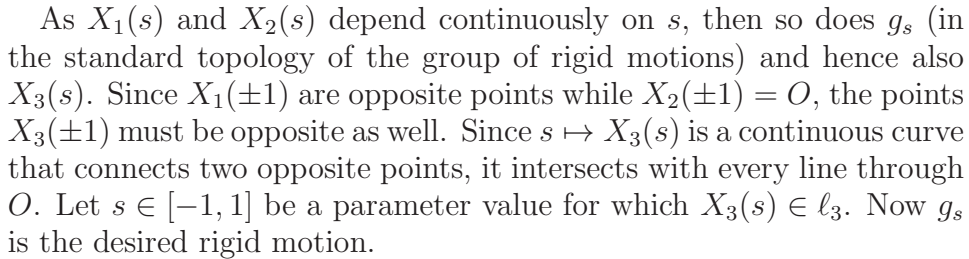
Proposition 3.4. *Let $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^n$ be lines through the origin O . The following statements are equivalent*

- (1) $\dim(\ell_1 + \ell_2 + \ell_3) = 2$
- (2) *For every triple of points $z_1, z_2, z_3 \in \mathbb{R}^2$ there exists a rigid motion g of \mathbb{R}^n with $g(z_j) \in \ell_j$ for each $j = 1, 2, 3$.*

Proof. (1) \Rightarrow (2). Since the group of rigid motions is transitive on the 2-planes in \mathbb{R}^n , we may assume that z_1, z_2 and z_3 belong to the subspace spanned by the lines. This reduces the proof to the case $n = 2$.

We shall assume the z_j are distinct as otherwise the result is easily seen. Furthermore, as at most two of the lines are identical, let us assume that $\ell_1 \neq \ell_2$. Let d denote the distance between z_1 and z_2 , and consider the set \mathfrak{X} of pairs (X_1, X_2) of points $X_1 \in \ell_1$ and $X_2 \in \ell_2$ with distance d from each other. Let D_1 be a point on ℓ_1 with distance d to the origin, then (D_1, O) and $(-D_1, O)$ belong to \mathfrak{X} , and it follows from the geometry that we can connect these points by a continuous curve $s \mapsto (X_1(s), X_2(s))$ in \mathfrak{X} , say with $s \in [-1, 1]$. For example, we can arrange that first $X_1(s)$ moves from $-D_1$ to O along ℓ_1 , while at the same time $X_2(s)$ moves along ℓ_2 at distance d from $X_1(s)$. Then $X_2(s)$ moves from O to a point $D_2 \in \ell_2$ at distance d from O . After that, $X_1(s)$ moves from O to D_1 , while $X_2(s)$ moves back from D_2 to O .

When s passes through the interval $[-1, 1]$, the line segment from $X_1(s)$ to $X_2(s)$ slides with its endpoints on the two lines. We define $X_3(s)$ such that the three points form a triangle congruent to the one formed by z_1, z_2 and z_3 . In other words, for each $s \in [-1, 1]$ there exists a unique rigid motion g_s of \mathbb{R}^n for which $g_s(z_1) = X_1(s)$ and $g_s(z_2) = X_2(s)$. We let $X_3(s) = g_s(z_3)$. See the following figure.



Let z_1, z_2, z_3 be an arbitrary triple of distinct points located on a common affine line ℓ , and let g be a rigid motion which brings these points into the ℓ_j . Then O can be one of the points $g(z_j)$, or not. In the first case, say if $g(z_1) = O$, it follows that ℓ_2 and ℓ_3 are both equal to $g(\ell)$, since each of these lines have two points in common with $g(\ell)$.

Hence $\dim(\ell_1 + \ell_2 + \ell_3) \leq 2$. In the second case, the line $g(\ell)$ together with O spans a 2-dimensional subspace of \mathbb{R}^n , which contains all the lines ℓ_j . Hence again $\dim(\ell_1 + \ell_2 + \ell_3) \leq 2$. \square

We proceed with the proof of Theorem 3.2.

Proof. Note that $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ are locally isomorphic to $\mathrm{SO}_e(2, 1)$ and $\mathrm{SO}_e(3, 1)$, respectively. The centers of $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ belong to K , and hence $G = KAH$ will hold for the triple spaces of these groups if and only if it holds for the triple spaces of their adjoint groups. Thus it suffices to consider $G_0 = \mathrm{SO}(n, 1)$ with $n \geq 2$.

The elements in $\mathfrak{so}(n, 1)$ have the form

$$(3.2) \quad X = \begin{pmatrix} A & b \\ b^t & 0 \end{pmatrix}$$

where $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$, and \mathfrak{so}_0 consists of the elements with $A = 0$.

Assume first that $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$ is 2-dimensional. By transitivity of the action of $K_0 = \mathrm{SO}(n)$ on the 2-dimensional subspaces of \mathbb{R}^n we may assume that $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$ consists of the matrices X as above with $A = 0$ and b non-zero only in the last two coordinates. Hence $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$ is contained in the $\mathfrak{so}(2, 1)$ -subalgebra in the lower right corner of $\mathfrak{so}(n, 1)$. It follows that $\exp(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3).z_0$ is a 2-dimensional totally geodesic submanifold of Z_0 .

Let $z_1, z_2, z_3 \in Z_0$ be given. Every triple of points in Z_0 belongs to a 2-dimensional totally geodesic submanifold Z'_0 of Z_0 . For example, in the model of Z_0 as a one-sheeted hyperboloid in \mathbb{R}^{n+1} , we can obtain Z'_0 as the intersection of Z_0 with a 3-dimensional subspace of \mathbb{R}^{n+1} containing the three points. Since G_0 is transitive on geodesic submanifolds, we may assume that z_1, z_2, z_3 are contained in the submanifold generated by $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$. We have thus essentially reduced to the case $n = 2$, and shall assume $n = 2$ from now on.

We proceed exactly as in the Euclidean case and produce a pair of points $X_1(s)$ and $X_2(s)$ on the geodesic lines $\exp(\mathfrak{a}_1).z_0$ and $\exp(\mathfrak{a}_2).z_0$, respectively. The two points are chosen so that they have the same non-Euclidean distance from each other as z_1 and z_2 , and they depend continuously on $s \in [-1, 1]$. Moreover, $X_1(-1)$ and $X_1(1)$ are symmetric with respect to z_0 , while $X_2(-1) = X_2(1) = z_0$. As Z_0 is two-point homogeneous, there exists for each $s \in [-1, 1]$ a unique isometry $g_s \in G_0$ such that $g_s(z_j) = X_j(s)$ for $j = 1, 2$. As before, a value of s , where the continuous curve $s \mapsto g_s(z_3)$ intersects $\exp(\mathfrak{a}_3)$, produces the desired isometry g_s of Property 3.3. Hence $G = KAH$.

We return to the case $n \geq 2$ and assume conversely that $G = KAH$. It follows from Lemma 3.1 that $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) > 1$. We want to exclude $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 3$. Again we follow the Euclidean proof and select an arbitrary triple of distinct points z_1, z_2, z_3 on a single geodesic γ in Z_0 . Then there is $g \in G_0$ such that $gz_j = y_j$ for some $y_j \in \exp(\mathfrak{a}_j).z_0$, for $j = 1, 2, 3$. If one of the y_j 's, say y_1 , is z_0 , then $\exp(\mathfrak{a}_2).z_0 = \exp(\mathfrak{a}_3).z_0 = g(\gamma)$ and hence $\mathfrak{a}_2 = \mathfrak{a}_3$. Otherwise, the geodesic $g(\gamma)$ is contained, together with O , in a 2-dimensional totally geodesic submanifold of Z_0 . This submanifold necessarily contains the geodesic $\exp(\mathfrak{a}_j).z_0$ for each j . Hence $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) \leq 2$. \square

4. UNIQUENESS

If G/H is a homogeneous space of polar type, so that every element $g \in G$ allows a decomposition $g = kah$, it is of interest to know to which extend the components in this decomposition are unique. An obvious non-uniqueness is caused by the normalizer $N_{K \cap H}(\mathfrak{a})$ of \mathfrak{a} in $K \cap H$, which acts on A by conjugation. In the case of a symmetric space, it is known (see [6], Prop. 7.1.3) that the A component of every $g \in G$ is unique up to such conjugation. For our current triple spaces the description of which elements in A generate the same $K \times H$ orbit appears to be more complicated, unless $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$.

Theorem 4.1. *Let G/H be the triple space with G_0 as in (1.1), and let \mathfrak{a} be as in (2.1) with $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$. Let $a = (a_1, a_2, a_3) \in A$ with $a_1 \neq a_2$ and let $a' = (a'_1, a'_2, a'_3) \in A$. Then $KaH = Ka'H$ if and only if a and a' are conjugate by $N_{K \cap H}(\mathfrak{a})$.*

We first determine explicitly which pairs of elements $a, a' \in A$ are $N_{K \cap H}(\mathfrak{a})$ -conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$.

Lemma 4.2. *Let \mathfrak{a} be as above. Then $a, a' \in A$ are conjugate by $N_{K \cap H}(\mathfrak{a})$ if and only if*

- (1) $(a'_1, a'_2) = (a_1, a_2)^{\pm 1}$ and $a'_3 = a_3^{\pm 1}$ if $n > 2$
- (2) $(a'_1, a'_2, a'_3) = (a_1, a_2, a_3)^{\pm 1}$ if $n = 2$.

Proof. The normalizer $N_{K \cap H}(\mathfrak{a})$ consists of all the diagonal elements $k = (k_0, k_0, k_0) \in G$ for which

$$k_0 \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_2) \cap N_{K_0}(\mathfrak{a}_3).$$

As elements $a_j, a'_j \in A_j$ are $N_{K_0}(\mathfrak{a}_j)$ -conjugate if and only if $a'_j = a_j^{\pm 1}$, only the pairs mentioned under (1) can be conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2$.

Let $\delta, \epsilon = \pm 1$. For the groups in (1.1) the adjoint representation is surjective $K_0 \rightarrow \mathrm{SO}(\mathfrak{s}_0)$. If $n > 2$ then there exists a transformation in $\mathrm{SO}(\mathfrak{s}_0)$ which acts by δ on $\mathfrak{a}_1 = \mathfrak{a}_2$ and by ϵ on \mathfrak{a}_3 . Its preimages in K_0

conjugate (a_1, a_2, a_3) to $(a_1^\delta, a_2^\delta, a_3^\epsilon)$. When $n = 2$ such a transformation exists if and only if $\delta = \epsilon$. The lemma follows. \square

The following lemmas are used in the proof of Theorem 4.1. Here G_0 can be any real reductive group with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$.

Lemma 4.3. *Let $X, U \in \mathfrak{s}_0$. Then $\exp X \exp U \exp X \in \exp \mathfrak{s}_0$.*

Proof. Let θ denote the Cartan involution and note that the product $\exp(tX) \exp(tU) \exp(tX)$ belongs to $S = \{g \in G_0 \mid \theta(g) = g^{-1}\}$ for all $t \in [0, 1]$. It is easily seen that $k \exp Y \in S$ implies $k^2 = e$ for $k \in K_0$ and $Y \in \mathfrak{s}_0$, and since e is isolated in the set of elements of order 2 it follows that $\exp \mathfrak{s}_0$ is the identity component of S . Hence $\exp X \exp U \exp X \in \exp \mathfrak{s}_0$. \square

Lemma 4.4. *Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be a one-dimensional subspace and let $A_0 = \exp \mathfrak{a}_0$.*

- (1) *If $g \in \exp \mathfrak{s}_0$ and $ga_0 \in a'_0 K_0$ for some $a_0, a'_0 \in A_0$, then $g = a'_0 a_0^{-1}$.*
- (2) *If $g \in G_0$ and $ga_1, ga_2 \in A_0 K_0$ for some $a_1, a_2 \in A_0$ with $a_1 \neq a_2$ then $g \in N_{K_0}(\mathfrak{a}_0) A_0$.*

Proof. (1) It follows from $ga_0 \in a'_0 K_0$ that $a_0 ga_0 \in a_0 a'_0 K_0$. Since $a_0 ga_0 \in \exp \mathfrak{s}_0$ by Lemma 4.3, it follows from uniqueness of the Cartan decomposition that $a_0 ga_0 = a_0 a'_0$ and thus $g = a'_0 a_0^{-1}$.

(2) Put $z_0 = eK_0$, then $A_0.z_0$ is a geodesic in G_0/K_0 . Since g maps two distinct points on $A_0.z_0$ into $A_0.z_0$, it maps the entire geodesic onto itself, and hence so does g^{-1} . In particular $g^{-1}.z_0 \in A_0 K_0$, that is, $g = k_0 a_0$ for some $k_0 \in K_0$, $a_0 \in A_0$. It follows for all $a \in A_0$ that

$$k_0 a k_0^{-1} = g a_0^{-1} a k_0^{-1} \in g A_0 K_0 = A_0 K_0.$$

As $k_0 a k_0^{-1} \in \exp \mathfrak{s}_0$, uniqueness of the Cartan decomposition implies $k_0 a k_0^{-1} \in A_0$, i.e. $k_0 \in N_{K_0}(\mathfrak{a}_0)$. \square

Lemma 4.5. *Let $\mathfrak{a}_1, \mathfrak{a}_3 \subset \mathfrak{s}_0$ be one-dimensional subspaces with $\mathfrak{a}_1 \perp \mathfrak{a}_3$ and let $A_1 = \exp \mathfrak{a}_1$, $A_3 = \exp \mathfrak{a}_3$. If $g \in N_{K_0}(\mathfrak{a}_1) A_1$ and $ga_3 \in a'_3 K_0$ for some $a_3, a'_3 \in A_3$, not both equal to e , then $g \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_3)$.*

Proof. We may assume $a'_3 \neq e$, as otherwise we interchange it with a_3 and replace g by g^{-1} . We consider the geodesic triangle in G_0/K_0 formed by the geodesics

$$L_1 := A_1.z_0, \quad L_2 := A_3.z_0, \quad L_3 := gA_3.z_0.$$

The vertices are

$$D_3 := z_0, \quad D_2 := g.z_0, \quad D_1 := ga_3.z_0 = a'_3.z_0.$$

As L_1 and L_2 intersect orthogonally, angle D_3 is right. The isometry g maps L_1 to itself and L_2 to L_3 . Hence L_1 and L_3 also intersect orthogonally and angle D_2 is right. As the sectional curvature of G_0/K_0 is ≤ 0 , it is impossible for a proper triangle to have two right angles. As $L_1 \neq L_2$ and $D_3 \neq D_1$ we conclude $D_3 = D_2$ and $L_3 = L_2$. It follows that $g \in K_0$ and by Lemma 4.4 (2) that $g \in N_{K_0}(\mathfrak{a}_3)$. \square

Proof of Theorem 4.1. Assume $KaH = Ka'H$. Then $Kah = Ka'$ for some $h = (g, g, g) \in H$. Applying Lemma 4.4 (2) to the first two coordinates of $Kah = Ka'$ we conclude that $g \in N_{K_0}(\mathfrak{a}_1)A_1$.

If a'_3 and a_3 are not both e , we can apply Lemma 4.5 to the last coordinate and conclude $g \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_3)$. Hence $h \in N_{K \cap H}(\mathfrak{a})$, and we conclude that $a' = h^{-1}ah$.

If $a'_3 = a_3 = e$ it follows from the third coordinate that $g \in K_0$. Hence $g \in N_{K_0}(\mathfrak{a}_1)$ and $a' = a$ or $a' = a^{-1}$. \square

Remark 4.6. When $\dim \mathfrak{s}_0 = 2$ the assumption in Theorem 4.1 and Lemmas 4.2, 4.5, that $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$, can be relaxed to $\mathfrak{a}_1 = \mathfrak{a}_2 \neq \mathfrak{a}_3$ with unchanged conclusions. This follows from the fact that in a two dimensional space the only proper orthogonal transformations which normalize a one-dimensional subspace are $\pm I$. Hence $N_{K_0}(\mathfrak{a}_1) = N_{K_0}(\mathfrak{a}_3)$ in this case.

5. A FORMULA FOR THE INVARIANT MEASURE

In a situation where there is uniqueness (up to some well-described isomorphism), it is of interest to explicitly determine the invariant measure with respect to the KAH -coordinates.

For any triple space G/H of a unimodular Lie group G_0 we note that the map

$$(5.1) \quad G_0 \times G_0 \rightarrow G/H, \quad (g_1, g_2) \mapsto (g_1, g_2, e)H$$

is a $G_0 \times G_0$ -equivariant diffeomorphism. Accordingly the invariant measure on G/H identifies with the Haar measure on $G_0 \times G_0$.

For $G_0 = \mathrm{SO}_e(n, 1)$ we define $X \in \mathfrak{so}(n, 1)$ by (3.2) with $A = 0$ and $b = e_n$, and $Y \in \mathfrak{so}(n, 1)$ similarly with $A = 0$ and $b = e_1$. Let $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathbb{R}X$ and $\mathfrak{a}_3 = \mathbb{R}Y$, then $\mathfrak{a}_3 \perp \mathfrak{a}_1$. Let

$$a_t = \exp(tX) \in A_1 = A_2, \quad b_s = \exp(sY) \in A_3.$$

Lemma 5.1. *Let G/H be the triple space of $G_0 = \mathrm{SO}_e(n, 1)$ and let $\mathfrak{a}_1 = \mathfrak{a}_2$ and \mathfrak{a}_3 be as above. Consider the polar coordinates*

$$(5.2) \quad K \times \mathbb{R}^3 \ni (k, t_1, t_2, s) \mapsto (k_1 a_{t_1}, k_2 a_{t_2}, k_3 b_s)H$$

on G/H . The invariant measure dz of G/H can be normalized so that in these coordinates

$$(5.3) \quad dz = J(t_1, t_2, s) dk dt_1 dt_2 ds$$

where dk is Haar measure, dt_1, dt_2, ds Lebesgue measure, and where

$$J(t_1, t_2, s) = |\sinh^{n-1}(t_1 - t_2) \sinh^{n-2}(s) \cosh(s)|.$$

Proof. On $G_0 \times G_0$ we use the formula (see [6], Thm. 8.1.1) for integration in KAH coordinates for the symmetric space $G_0 \times G_0 / \text{diag}(G_0) = G_0$. The map

$$(K_0 \times K_0) \times A_0 \times G_0 \rightarrow G_0 \times G_0$$

defined by

$$(k, a_t, g) \mapsto (k_1 a_{t/2} g, k_2 a_{-t/2} g)$$

is a parametrization (up to the sign of t), and the Haar measure on $G_0 \times G_0$ writes as

$$(5.4) \quad |\sinh^{n-1}(t)| dk_1 dk_2 dt dg.$$

Further we decompose the diagonal copy of G_0 by means of the HAK coordinates for the symmetric space $G_0 / (\text{SO}(n-1) \times A_1)$, where $\text{SO}(n-1)$ is located in the upper left corner of G_0 . Note that the subgroup A_3 serves as the ‘ A ’ in this decomposition. In the coordinates

$$K_0 \times A_3 \times \text{SO}(n-1) \times A_1 \rightarrow G_0, \quad (k_3, b_s, m, a_u) \mapsto a_u m b_s k_3$$

we obtain (again using [6], Thm. 8.1.1),

$$(5.5) \quad dg = |\sinh^{n-2}(s) \cosh(s)| dk_3 db_s dm du.$$

Combining (5.4) and (5.5), we have the coordinates

$$(k_1 a_{u+t/2} m b_s k_3, k_2 a_{u-t/2} m b_s k_3)$$

on $G_0 \times G_0$, with Jacobian $|\sinh^{n-1}(t) \sinh^{n-2}(s) \cosh(s)|$. As the subgroup $\text{SO}(n-1)$ centralizes A_1 , the integration over m is swallowed by the integrations over k_1 and k_2 . Changing coordinates u, t to $t_1 = u + t/2$ and $t_2 = u - t/2$ we find $t = t_1 - t_2$.

Finally we apply (5.1) so that the above coordinates correspond to

$$(k_1, k_2, k_3)(a_{t_1}, a_{t_2}, b_{-s}) \text{diag}(G_0)$$

This proves (5.3). □

6. SPHERICAL DECOMPOSITION

A decomposition of \mathfrak{g} of the form

$$(6.1) \quad \mathfrak{g} = \mathfrak{p} + \mathfrak{h}$$

with \mathfrak{p} a minimal parabolic subalgebra is said to be a *spherical* decomposition. If such a decomposition exists, then the homogeneous space G/H is said to be of *spherical type* (see [5]).

Note that with $\mathfrak{g}_0 = \mathfrak{so}(n, 1)$ we have (see (6.4) and (6.5))

$$\dim \mathfrak{p} + \dim \mathfrak{h} - \dim \mathfrak{g} = \frac{1}{2}(n^2 - 5n + 6) \geq 0.$$

In particular spherical decompositions will be direct sums if $n = 2, 3$.

It was observed in [5] that the triple spaces for the groups considered in (1.1) are of spherical type. In the following we determine for each n all the minimal parabolic subalgebras \mathfrak{p} for which (6.1) holds.

Proposition 6.1. *Let G_0 be one of the groups (1.1) and let $\mathfrak{p} = \mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3$ a minimal parabolic subalgebra. Then $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ holds if and only if \mathfrak{p}_1 , \mathfrak{p}_2 and \mathfrak{p}_3 are distinct.*

In particular, the triple space G/H is of spherical type for all groups G_0 in (1.1).

We prepare by the following lemma.

Lemma 6.2. *Let $U_1, U_2, U_3 \subset V$ be subspaces of a vector space V . Put*

$$U := U_1 \times U_2 \times U_3 \subset X := V \times V \times V,$$

and $Y := \text{diag}(V) \subset X$. Then $X = U + Y$ if and only if

$$(6.2) \quad V = U_1 + (U_2 \cap U_3) = U_2 + (U_3 \cap U_1) = U_3 + (U_1 \cap U_2).$$

Proof. Assume first that $X = U + Y$ and let $v \in V$ be given. Writing

$$(v, 0, 0) = (u_1, u_2, u_3) + \text{diag}(w)$$

we see that $w = -u_2 = -u_3 \in U_2 \cap U_3$, and hence $v = u_1 + w \in U_1 + (U_2 \cap U_3)$. The other two statements in (6.2) follow similarly.

Conversely, we assume (6.2) and let $x = (x_1, x_2, x_3) \in X$ be given. We decompose x_1 , x_2 and x_3 according to the three decompositions in (6.2), that is,

$$\begin{aligned} x_1 &= u_1 + t_1, & u_1 &\in U_1, \ t_1 \in U_2 \cap U_3 \\ x_2 &= u_2 + t_2, & u_2 &\in U_2, \ t_2 \in U_3 \cap U_1 \\ x_3 &= u_3 + t_3, & u_3 &\in U_3, \ t_3 \in U_1 \cap U_2. \end{aligned}$$

Then

$$x = (u_1 - t_2 - t_3, u_2 - t_1 - t_3, u_3 - t_1 - t_2) + \text{diag}(t_1 + t_2 + t_3)$$

is a decomposition of the desired form $U + Y$. \square

Remark 6.3. In fact, it is easily seen that any two of the decompositions of V in (6.2) together imply the third.

Proof of Proposition 6.1. It suffices to consider $G_0 = \mathrm{SO}_0(n, 1)$ because of the local isomorphisms.

If for example $\mathfrak{p}_1 = \mathfrak{p}_2$ then $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3) = \mathfrak{p}_1$. Hence $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3)$ is proper in \mathfrak{g}_0 and it follows from Lemma 6.2 that $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ fails to hold. This implies one direction of the first statement.

For the other direction it follows from Lemma 6.2 that it suffices to prove

$$\mathfrak{g}_0 = \mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3)$$

for all triples of distinct parabolics in $\mathfrak{so}(n, 1)$. We shall do this by proving

$$(6.3) \quad \dim \mathfrak{g}_0 = \dim \mathfrak{p}_1 + \dim(\mathfrak{p}_2 \cap \mathfrak{p}_3) - \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3).$$

We find

$$(6.4) \quad \dim \mathfrak{g}_0 = \dim \mathfrak{so}(n, 1) = \frac{1}{2}(n^2 + n),$$

and claim that

$$(6.5) \quad \dim \mathfrak{p}_1 = \frac{1}{2}(n^2 - n + 2)$$

$$(6.6) \quad \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2) = \frac{1}{2}(n^2 - 3n + 4)$$

$$(6.7) \quad \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = \frac{1}{2}(n^2 - 5n + 6).$$

The equations (6.4)-(6.7) imply (6.3).

The parabolic subalgebras \mathfrak{p} of $\mathfrak{so}(n, 1)$ are the normalizers of the isotropic lines in \mathbb{R}^{n+1} , that is, the one-dimensional subspaces of the form $L_q = \mathbb{R}(q, 1)$ where $q \in \mathbb{R}^n$ with $\|q\| = 1$.

Recall that all elements in $\mathfrak{so}(n, 1)$ have the form (3.2) with $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$. It follows that $X \in \mathfrak{p}$ if and only if

$$(6.8) \quad Aq + b = (b \cdot q)q.$$

Let us prove (6.5). Let q_1 be the unit vector such that \mathfrak{p}_1 is the stabilizer of L_{q_1} , and extend q_1 to a basis q_1, \dots, q_n for \mathbb{R}^n . For $b \in \mathbb{R}^n$ we let $x_1 = (b \cdot q_1)q_1 - b$ and we observe that $x_1 \cdot q_1 = 0$. According to (6.8) the matrix X of (3.2) belongs to \mathfrak{p}_1 if and only if $Aq_1 = x_1$. In order to satisfy that we can define an $n \times n$ matrix A by

$$(6.9) \quad Aq_i \cdot q_j := \begin{cases} x_1 \cdot q_j & \text{for } i = 1 \\ -x_1 \cdot q_i & \text{for } j = 1 \\ a_{ij} & \text{for } i, j > 1 \end{cases}$$

with arbitrary antisymmetric entries in the last line. Then $A \in \mathfrak{so}(n)$ and $Aq_1 = x_1$. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-1) = \frac{1}{2}(n-1)(n-2),$$

and hence $\dim \mathfrak{p}_1 = n + \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(n^2 - n + 2)$ as asserted.

Next we prove (6.6). Let q_1, q_2 be the unit vectors such that \mathfrak{p}_i is the stabilizer of L_{q_i} . By assumption $q_1 \neq q_2$. For the element X of (3.2) to be in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ we need that (6.8) is satisfied in both cases, that is,

$$(6.10) \quad Aq_i = x_i, \quad (i = 1, 2).$$

where $x_i = (b \cdot q_i)q_i - b$. Now

$$x_2 \cdot q_1 + x_1 \cdot q_2 = (q_1 \cdot q_2 - 1)(b \cdot (q_1 + q_2)).$$

Note that $q_1 \cdot q_2 < 1$ since $q_1 \neq q_2$. As $A \in \mathfrak{so}(n)$ we conclude that

$$b \cdot (q_1 + q_2) = 0$$

since otherwise (6.10) would lead to contradiction.

Conversely, let $b \in \mathbb{R}^n$ be such that $b \cdot (q_1 + q_2) = 0$ and define x_1, x_2 by $x_i = (b \cdot q_i)q_i - b$. Then $x_i \cdot q_j = -x_j \cdot q_i$ for all pairs $i, j \leq 1, 2$. We extend q_1, q_2 to a basis and define an $n \times n$ matrix A by

$$(6.11) \quad Aq_i \cdot q_j = \begin{cases} x_i \cdot q_j & \text{for } i = 1, 2 \\ -x_j \cdot q_i & \text{for } j = 1, 2 \\ a_{ij} & \text{for } i, j > 2 \end{cases}$$

with arbitrary antisymmetric entries in the last line. Then $A \in \mathfrak{so}(n)$ and (6.10) holds. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-2) = \frac{1}{2}(n-2)(n-3)$$

and hence $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2) = n - 1 + \frac{1}{2}(n-2)(n-3) = \frac{1}{2}(n^2 - 3n + 4)$ as asserted.

Finally, to prove (6.7) assume that X in (3.2) belongs to $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$. As above, it follows that

$$b \cdot (q_1 + q_2) = b \cdot (q_1 + q_3) = b \cdot (q_2 + q_3) = 0$$

which implies that $b \cdot q_i = 0$ for $i = 1, 2, 3$. If this is satisfied by b , the condition (6.8) simplifies to

$$(6.12) \quad Aq_i = -b, \quad i = 1, 2, 3.$$

We first assume that q_1, q_2, q_3 are linearly independent and extend to a basis as before. Given $b \in \mathbb{R}^n$ such that $b \cdot q_i = 0$ for $i = 1, 2, 3$ we

define A by

$$(6.13) \quad Aq_i \cdot q_j = \begin{cases} -b \cdot q_j & \text{for } i = 1, 2, 3 \\ b \cdot q_i & \text{for } j = 1, 2, 3 \\ a_{ij} & \text{for } i, j > 3 \end{cases}$$

with arbitrary antisymmetric entries in the last line. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-3) = \frac{1}{2}(n-3)(n-4)$$

and hence $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = n-3 + \frac{1}{2}(n-3)(n-4) = \frac{1}{2}(n^2 - 5n + 6)$ as asserted.

Next we assume linear dependence of q_1, q_2, q_3 . This implies a further obstruction to b . In fact, let $\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = 0$ be a non-trivial relation, then it follows from (6.12) that $(\lambda_1 + \lambda_2 + \lambda_3)b = 0$. Since q_1, q_2, q_3 are assumed to be distinct unit vectors the sum of the λ 's cannot be zero, and we conclude that $b = 0$. Thus in this case the only freedom comes from the choice of A . That can be chosen arbitrarily from the annihilator in $\mathfrak{so}(n)$ of the space spanned by the three q 's. We obtain $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = \dim \mathfrak{so}(n-2) = \frac{1}{2}(n^2 - 5n + 6)$ as before. This concludes the proof of (6.7).

In particular, if $\mathfrak{a}_1, \mathfrak{a}_2$ and \mathfrak{a}_3 are all different, then $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ for every parabolic subalgebra \mathfrak{p} above $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$. Hence G/H is of spherical type. \square

Corollary 6.4. *There exists a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ for which both*

- (i) *the polar decomposition (3.1) is valid, and*
- (ii) *the spherical decomposition (6.1) is valid for all minimal parabolic subalgebras containing \mathfrak{a} .*

Proof. Let $\mathfrak{a}_j \subset \mathfrak{s}_0$ for $j = 1, 2, 3$ be mutually different and with a two-dimensional sum. It follows from Theorem 3.2 and Proposition 6.1 that $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ satisfies (i) and (ii). \square

Remark 6.5. The properties of a reductive homogeneous space G/H that it is of polar type, respectively of spherical type, appear to be closely related. However, the relation is not as strong as one might hope, because the conditions on \mathfrak{a} are different in Theorem 3.2 and Proposition 6.1. In particular, there exist maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{g}$ which fulfill (ii) but not (i), namely the 'most generic' ones, for which $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 3$.

7. INFINITESIMAL POLAR DECOMPOSITION

Here we consider an infinitesimal version of the polar decomposition $G = KAH$. Let G/H be a homogeneous space of a reductive group G , and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition.

Definition 7.1. *A decomposition of the form*

$$(7.1) \quad \mathfrak{s} = \text{Ad}(K \cap H)\mathfrak{a} + \mathfrak{s} \cap \mathfrak{h}$$

with an abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ is called a polar decomposition.

If there exists such a decomposition of \mathfrak{s} then we say that G/H is infinitesimally polar.

Here

$$\text{Ad}(K \cap H)\mathfrak{a} = \{\text{Ad}(k)X \mid k \in K \cap H, X \in \mathfrak{a}\}.$$

Note that this need not be a vector subspace of \mathfrak{s} .

If G/H is a symmetric space, then we can choose the Cartan decomposition so that \mathfrak{k} and \mathfrak{s} are stable under the involution σ that determines G/H . If $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ denotes the decomposition of \mathfrak{g} in $+1$ and -1 eigenspaces for σ , then $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{q} + \mathfrak{s} \cap \mathfrak{h}$. If furthermore \mathfrak{a}_q is a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$, then it is known that $\mathfrak{s} \cap \mathfrak{q} = \text{Ad}(K \cap H)\mathfrak{a}_q$ and hence (7.1) follows.

The following lemma suggests that there is a close connection between polar decomposability and infinitesimally polar decomposability.

Lemma 7.2. *Let G_0 be one of groups (1.1) and let $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$. Then the infinitesimal polar decomposition (7.1) holds if and only if $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 2$.*

Proof. For the triple spaces, the polar decomposition (7.1) interprets to the statement that for every triple of points $Z_1, Z_2, Z_3 \in \mathfrak{s}_0$ there exist $k \in K_0$, $T \in \mathfrak{s}_0$ and $X_j \in \mathfrak{a}_j$ ($j = 1, 2, 3$) such that $Z_j = \text{Ad}(k)X_j + T$. As the maps $X \mapsto \text{Ad}(k)X + T$ with $k \in K_0$ and $T \in \mathfrak{s}_0$ are exactly the rigid motions of \mathfrak{s}_0 , this lemma is precisely the content of Proposition 3.4. \square

Combining the lemma with Theorem 3.2 we see that for our triple spaces the infinitesimal polar decomposition holds with a given \mathfrak{a} if and only if the global polar decomposition $G = KAH$ holds for the corresponding $A = \exp \mathfrak{a}$.

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